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Note

# On the inequalities of Ky Fan, Wang–Wang and Alzer

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We denote by $A$, $B$ and $H$ (or $A(a)$, $G(a)$ and $H(a)$) the unweighted arithmetic, geometric and harmonic means of the real numbers $a_1, \dots, a_n$ with $a = (a_1, \dots, a_n) \in \mathbb{R}_+^n$. Let $1 = (1, \dots, 1)$, $A^- = A(1-a)$, $G^- = G(1-a)$, $H^- = H(1-a)$ with $0 < a_i < 1$, $i = 1, \dots, n$, and $A^+ = A(1+a)$, $G^+ = G(1+a)$, $H^+ = H(1+a)$ with $a_i > 0$, $i = 1, \dots, n$. The fundamental inequalities

$$H \leqslant G \leqslant A \tag{1}$$

can be considered as the separation of arithmetic means from harmonic by geometric means. The same goes for

$$\frac{H}{H^-} \leqslant \frac{G}{G^-} \leqslant \frac{A}{A^-}, \tag{2}$$

$$\frac{H}{H^+} \leqslant \frac{G}{G^+} \leqslant \frac{A}{A^+}, \tag{3}$$

and

$$\frac{1}{H^-} - \frac{1}{H} \leqslant \frac{1}{G^-} - \frac{1}{G} \leqslant \frac{1}{A^-} - \frac{1}{A}, \tag{4}$$

with $a_i \in (0, 1/2]$, $i = 1, \dots, n$.

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The inequalities (2) are due to Ky Fan (right-hand) and W.-L. Wang and P.-F. Wang (left-hand), while (4) has been proved by H. Alzer (see [1]). For the inequalities (3) we refer to the paper [2].

In this note we shall give their simple and short proofs as well as the proofs of the following inequalities.

**Theorem 1.** *If $a \in \mathbb{R}_+^n$, then*

$$\frac{1}{H^+} - \frac{1}{H} \leqslant \frac{1}{G^+} - \frac{1}{G} \leqslant \frac{1}{A^+} - \frac{1}{A} \tag{5}$$

*with the equalities if and only if $a_1 = \cdots = a_n$.*

**Proof.** By the well-known inequality (8) and $G \leqslant A$ we have

$$\frac{1}{G^+} - \frac{1}{G} \leqslant \frac{1}{1+G} - \frac{1}{G} \leqslant \frac{-1}{(1+A)A} = \frac{1}{A^+} - \frac{1}{A}.$$

According to (3) (left-hand) and $H^+ < G^+$ (the case of $H^+ = G^+$ is trivial) we get

$$\frac{HG}{H^+G^+} \leqslant \frac{H}{H^+} \leqslant \frac{H}{H^+}\frac{G/H-1}{G^+/H^+-1} = \frac{G-H}{G^+-H^+};$$

i.e.,

$$\frac{1}{H^+} - \frac{1}{G^+} \leqslant \frac{1}{H} - \frac{1}{G}. \qquad \square$$

The inequality

$$\frac{1}{H^+} - \frac{1}{H} \leqslant \frac{1}{A^+} - \frac{1}{A}$$

is equivalent to Jensen's inequality for strictly convex function $f(x) = 1/x - 1/(1+x)$, $x > 0$, such that the sign of equality holds if and only if $a_1 = \cdots = a_n$.

**Proof of inequalities (2).** Substituting $(1-a_i)/a_i$ with $x_i > 1$ we see that the Ky Fan inequality is equivalent to $H(1+x) \leqslant 1 + G(x)$, i.e.,

$$H^+ \leqslant 1 + G, \tag{6}$$

while the Wang–Wang inequality is equivalent to

$$G - 1 \leqslant A - \frac{G}{H}. \tag{7}$$

Let $n > 1$, and fix an $\alpha > 1$. Each of the functions

$$f(x) = \frac{1}{H(1+x)} = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{1+x_i}$$

and

$$w(x) = A(x) - \frac{\alpha}{H(x)} = \frac{1}{n}\sum_{i=1}^{n}\left(x_i - \frac{\alpha}{x_i}\right)$$

attains its absolute minimum in the set

$$S = \left\{x \in \mathbb{R}_+^n \mid G(x) = \alpha,\ 1 \leqslant x_i \leqslant \alpha^n,\ i = 1, \ldots, n\right\}.$$

Let $b = (b_1, \ldots, b_n) \in S$ and $c = (c_1, \ldots, c_n) \in S$ be points of minimum of $f(x)$ and $w(x)$, respectively. If $b \neq (\alpha, \ldots, \alpha)$ and $c \neq (\alpha, \ldots, \alpha)$ we can assume that $b_1 < \alpha < b_2$ and $c_1 < \alpha < c_2$ ($f(x)$ and $w(x)$ are symmetric under the permutation of $x_1, \ldots, x_n$). Then

$$\bar{b} = \left(\sqrt{b_1 b_2}, \sqrt{b_1 b_2}, b_3, \ldots, b_n\right) \in S,$$
$$\bar{c} = \left(\sqrt{c_1 c_2}, \sqrt{c_1 c_2}, c_3, \ldots, c_n\right) \in S,$$
$$f(b) - f(\bar{b}) = \frac{(\sqrt{b_1} - \sqrt{b_2})^2(\sqrt{b_1 b_2} - 1)}{n(1 + b_1)(1 + b_2)(1 + \sqrt{b_1 b_2})} > 0,$$

and

$$w(c) - w(\bar{c}) = \frac{(\sqrt{c_1} - \sqrt{c_2})^2(c_1 c_2 - \alpha)}{n c_1 c_2} > 0$$

(since $c_1 c_2 \geqslant c_2 > \alpha$), which is in contradiction with the definition of $b$ and $c$. Therefore, we must have for all $x \in S$

$$f(x) \geqslant f(\alpha, \ldots, \alpha) = \frac{1}{1 + \alpha}$$

and

$$w(x) \geqslant w(\alpha, \ldots, \alpha) = \alpha - 1.$$

This yields (6) and (7). □

**Proof of inequalities (3).** By the concavity of $G(x) = \prod_{i=1}^{n} x_i^{1/n}$ on $\mathbb{R}_+^n$ we get

$$\frac{G(1) + G(a)}{2} \leqslant G\left(\frac{1 + a}{2}\right);$$

i.e.,

$$1 + G \leqslant G^+. \tag{8}$$

Thus,

$$\frac{G}{G^+} \leqslant \frac{G}{1 + G} \leqslant \frac{A}{1 + A} = \frac{A}{A^+}.$$

Also, the mapping $x \to -(\operatorname{grad} G(x))$ is monotonic on $\mathbb{R}_+^n$; i.e., for all $x, y \in \mathbb{R}_+^n$ holds

$$\left\langle \operatorname{grad} G(x) - \operatorname{grad} G(y), x - y \right\rangle \leqslant 0,$$

where $\langle \cdot, \cdot \rangle$ is the inner product in $\mathbb{R}^n$. Putting $x = a$ and $y = 1 + a$ we get

$$\frac{H}{H^+} \leqslant \frac{G}{G^+}. \qquad \square \tag{9}$$

Finally, we shall show that the Alzer inequalities (4) are the consequence of (1), (2) and

$$H \leqslant H^-, \qquad A \leqslant A^-, \qquad a_i \in \left(0, \frac{1}{2}\right].$$

**Proof of inequalities (4).**

$$\frac{1}{G^-} - \frac{1}{G} = \frac{1}{G}\left(\frac{G}{G^-} - 1\right) \geqslant \frac{1}{G}\left(\frac{H}{H^-} - 1\right) = \frac{H}{G}\frac{H - H^-}{HH^-} \geqslant \frac{1}{H^-} - \frac{1}{H},$$
$$\frac{1}{G^-} - \frac{1}{G} = \frac{1}{G}\left(\frac{G}{G^-} - 1\right) \leqslant \frac{1}{G}\left(\frac{A}{A^-} - 1\right) = \frac{A}{G}\frac{A - A^-}{AA^-} \leqslant \frac{1}{A^-} - \frac{1}{A}.$$

$\square$

## References

[1] H. Alzer, Inequalities for arithmetic, geometric and harmonic means, Bull. London Math. Soc. 22 (1990) 362–366.

[2] E. El-Neweihi, F. Proschan, Unified treatment of some inequalities among ratios of means, Proc. Amer. Math. Soc. 81 (1981) 388–390.